On Non-normal Graphic Types

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The list of 1040 magic squares includs a classification of each square according to its "graphic type". This is a reference to H.E. Dudeney's well known system for dividing the normal squares into 12 numbered types depending on the 12 different patterns in which the eight conjugate pairs, 1 & 16, 2 & 15, .., 8 & 9, are found to occur [Amusements in Mathematics, page 120]. Dudeney drew lines to indicate these patterns (hence the "graphic"), but they can be shown equally well using examples from among the 1040 squares, the conjugate pairs then being 1 & -1, 2 & -2, etc.:

```
    type 1         type 2         type 3
-8 -1  2  7    -8 -5  6  7    -8 -1  4  5
 4  5 -6 -3     5  8 -7 -6     6  3 -2 –7
-2 -7  8  1    -1 -4  3  2     7  2 -3 –6
 6  3 -4 -5     4  1 -2 -3    -5 -4  1  8

    type 4         type 5         type 6
-8 -5  6  7    -8 -5  8  5    -8 -7  7  8
 8  5 -6 -7     6  7 -6 -7     2  6 -6 -2
-2 -3  4  1    -2 -3  2  3     5 -3  3 -5
 2  3 -4 -1     4  1 -4 -1     1  4 -4 -1

    type 7         type 8         type 9
-7 -5  8  4    -8 -6  8  6    -8 -5  7  6
 1  5 -8  2    -1  7 -7  1     8  1 -3 -6
-1 -3  6 -2     5 -3  3 -5     2 -1  3 -4
 7  3 -6 -4     4  2 -4 -2    -2  5 -7  4

   type 10        type 11        type 12
-7 -5  5  7    -8  6 -2  4    -8 -6  8  6
 6  8 -6 -8     8 -4  2 -6     2  4 -2 -4
-2 -4  2  4     1 -5  7 -3     7 -3  1 -5
 3  1 -1 -3    -1  3 -7  5    -1  5 -7  3
```

An algebraic generalization of each type is easy to derive. The following are taken from my 1980 article "Magic Formulae":

```
       type 1                 type 2                 type 3
 a+b -a+d -c-d -b+c     a+b -c+d -a-d -b+c     a+b -b-c -a-d  c+d
-a-d  a-b  b+c -c+d     c-d -a-b  b-c  a+d    -b+c -a+b -c+d  a-d
 c+d  b-c -a-b  a-d    -a+d  b+c  a-b -c-d    -a+d  c-d  a-b  b-c
-b-c  c-d  a+d -a+b    -b-c  a-d  c+d -a+b    -c-d  a+d  b+c -a-b

       type 4                 type 5                 type 6
 a+b -c+d -a-d -b+c     a+b -b-c -a+b -b+c     a+b -a+d -a-d  a-b
-a-b  c-d  a+d  b-c     c-d -a+d -c-d  a+d     c+e -b-c  b-c  c-e
 a-b -c-d -a+d  b+c    -a-b  b+c  a-b  b-c    -c-e  b+c -b+c -c+e
-a+b  c+d  a-d -b-c    -c+d  a-d  c+d -a-d    -a-b  a-d  a+d -a+b
```

```
       type 7                 type 8                 type 9
  a   b-c -a-b   c    a+b -c+d -a-d -b+c     a+b -c+d -a-d -b+c
-a-d -b+c  a+b -c+d   a-b  -a   -c   b+c    -a-b   c    a   b-c
 a+d -b-c -a+b  c-d  -a-b   a    c   b-c     a-b  -c   -a   b+c
 -a   b+c  a-b  -c   -a+b  c-d  a+d -b-c    -a+b  c-d  a+d -b-c

       type 10                type 11                type 12
  a  -a+b -b-c   c    a+b -2a-c  a-b   c     a+b  a-b -2a-c   c
 a+d -a-b  b-c  c-d  -a-b  -c   -a+b  2a+c   b+c -b+c   a   -a-2c
-a-d  a-b  b+c -c+d   b+c   a   -b+c -a-2c  -a-b -a+b  -c   2a+c
 -a   a+b -b+c  -c   -b-c  a+2c  b-c  -a    -b-c  b-c  a+2c  -a
```

[Note that the magic constant in the above squares is zero; add k/4 to every cell to generalize squares having any magic constant, k.]

A natural question suggested by the above concerns non-normal squares: Do Dudeney's 12 types account for every possible pattern if squares using _any_ 8 conjugate pairs (i.e., pairs of equal sum) are allowed?

The answer has been known in Japan for more than 40 years. If the question were ever posed, I can find no trace of it in the magic square literature of the West.

The first non-Dudeney type square was discovered by Gakuho Abe in 1957. Abe was investigating a special kind of 7×7 square containing embedded 3×3 and 4×4 magic squares in opposite corners:

```
     40   5  30 | 34  16  12  38
  >  15  25  35 | 28  22  24  26
     20  45  10 | 13  37  39  11
     ---------------------------
     18  21  36 |  8  49  41   2
     32  29  14 |  1  44  46   9  <
     19  23  33 | 43   4   6  47
     31  27  17 | 48   3   7  42
```

The 4×4 squares were thus constructed using numbers in the range 1 to 49, and although non-normal, could yet include 8 conjugate pairs (1,49; 2,48; 3,47; 4,46; 6,44; 7,43; 8,42; 9,41, here). Abe noticed that in a few cases their distribution was non-Dudeney, as above. He went on to discover six non-Dudeney types that later appeared in his article "Yon Ho jin no taiikei no zensaku" or "Complete Counterpoint Models of Order Four Magic Squares."

Nineteen years later an interesting response appeared. In 1976 Tomiya Yokose published a systematic study of all possible graphic types in which he identified 30 in all, Dudeney's 12 included [Sugei no pazuru (= "Mathematical Puzzles"), No.92, Sept-Oct., Showa 51, pp 17-25]. In my humble opinion Yokose's work is a milestone in the development of the theory of 4×4 squares. As before, however, news of it seems never to have reached the West.

I am in debt to Nyr Indictor of New York State, and Guido de Mey of Brussels for translations of Abe's and Yokose's articles.

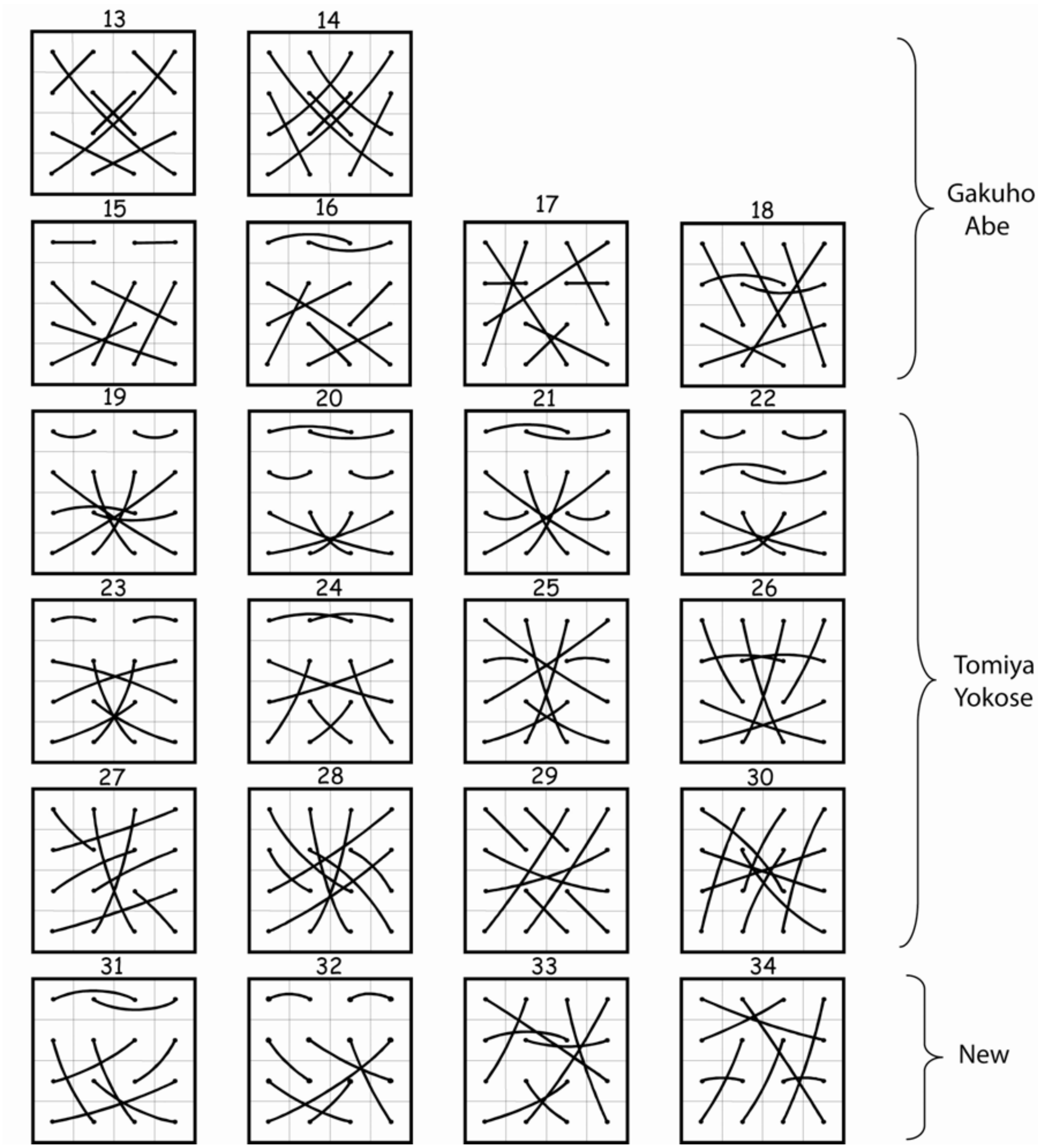

Figure 1. The 22 non-normal graphic types

Still later, Michael Schweitzer and I looked afresh at this question, during which we discovered four types that Yokose had missed; see Figure 1. A computer search was employed; it would be tedious to describe the method here. Following on from Dudeney's 12, I number the new types 13 to 34, an example of each (using the smallest integers possible) being as follows:

```
    type 13        type 14                                     --
  4 -9 -1  6     4 -1 -9  6                                     |
  9 -2 -8  1    -7  2  8 -3                                     |
 -7  8  2 -3     9 -8 -2  1                                     |
 -6  3  7 -4    -6  7  3 -4                                     |
                                                                |- Gakuho
    type 15        type 16        type 17       type 18         |   Abe
  7 -7 21-21     7 21 -7-21    -3-11 13  1    -9 -5  3 11       |
-11 -3  1 13    -5 -9 11  3    -7  7-21 21    21  7-21 -7       |
 -5 11 -9  3   -11  1 -3 13    -1  9  5-13   -13  9  5 -1       |
  9 -1-13  5     9-13 -1  5    11 -5  3 -9     1-11 13 -3       |
                                                               --

    type 19        type 20        type 21       type 22        --
  7 -7 -3  3     7 -3 -7  3    -6  2  8 -4     1 -1 -9  9       |
  2 -6 -4  8    -1  1  9 -9    -7  7  3 -3    -3  7  3 -7       |
 -1  9  1 -9     2 -4 -6  8     4 -8 -2  6     6 -8 -2  4       |
 -8  4  6 -2    -8  6  4 -2     9 -1 -9  1    -4  2  8 -6       |
                                                                |
    type 23        type 24        type 25       type 26         |
-16 16-14 14   -16-14 16 14    -2 11-13  4     8 13-11-10       |
 11 -2  4-13    13  8-10-11    16-16 14-14   -14-16 14 16       |- Tomiya
 13-10  8-11    11  4 -2-13    -4 -8 10  2     2 -8 10 -4       |  Yokose
 -8 -4  2 10    -8  2 -4 10   -10 13-11  8     4 11-13 -2       |
                                                                |
    type 27        type 28        type 29        type 30        |
 11  2-14  1    11-14  2  1   -11 -1  5  7     9 -7  3 -5       |
 -1-11  7  5    -7  9 -5  3     2 11  1-14   -14 11  1  2       |
 -7 -5  9  3    -1  7-11  5    14 -3 -9 -2    -2 -3 -9 14       |
 -3 14 -2 -9    -3 -2 14 -9    -5 -7  3  9     7 -1  5-11      --

    type 31        type 32       type 33         type 34       --
  7 -3 -9  5     7 -9 -3  5   -11 21-13  3    -5 -7 -1 13       |
 21-11  3-13    -7 -5 13 -1    -3  7  5 -9    -9  7  5 -3       |- New
 -7 13 -5 -1    21  3-11-13     1-21  9 11    11-21  9  1       |
-21  1 11  9   -21 11  1  9    13 -7 -1 -5     3 21-13-11      --
```

Note that squares in the same row above all use the same set of integers. In fact those types appearing in the same row are merely rearrangements of each other, a feature that is reflected in their algebraic generalizations, or formulas. Below follow algebraic formulas of types 13, 15, 19, 23, 27, and 31, which are those types appearing left in the above list. Formulas for the remaining types to the right of these employ the same algebraic terms rearranged in the same way as their corresponding integers. For example, the formula for type 14 is merely a rearrangement of that for type 13, as shown below, and the formulas for types 16, 17, 18 are again analogous rearrangements of the formula for type 15, and so on for the remaining types.

```
         type 13                          type 14
-a+c   -a-b-c  a+b-c  a+c         -a+c    a+b-c -a-b-c  a+c
 a+b+c  b-c    -b-c   -a-b+c       a-b-c -b+c    b+c    -a+b-c
 a-b-c  b+c    -b+c   -a+b-c       a+b+c -b-c    b-c    -a-b+c
-a-c    a-b+c -a+b+c  a-c         -a-c   -a+b+c  a-b+c  a-c
```

```
        type 15
  7a  -7a   21a  -21a
-11a  -3a     a   13a
 -5a  11a   -9a    3a
  9a   -a  -13a    5a

        type 19
 4a-b -3a-b  2a+b -3a+b
    b -5a+b  2a-b  3a-b
-2a+b  3a+b    -b  -a-b
-2a-b  5a-b -4a+b   a+b

        type 23
-5a-b  5a+b -5a+b  5a-b
 4a-b  -a+b   a+b -4a-b
 4a+b -3a-b  3a-b -4a+b
-3a+b  -a-b   a-b  3a+b

        type 27
 4a-b   a   -7a   2a+b
-2a-b -4a+b  2a-b 4a+b
-2a+b -4a-b  6a+b   -b
    b  7a    -a  -6a-b

        type 31
   7a  -3a -9a   5a
  21a -11a  3a -13a
  -7a  13a -5a   -a
 -21a    a 11a   9a
```

A careful look at 34 formulas shows that the sets of 16 algebraic terms they employ are just nine in number. This itself suggests a new (and more meaningful) method of classifying 4×4 magic squares according to which of these 9 classes it belongs:

Class	Types	Sets of terms in formulae (negatives excluded)							
1	1,2,3,4,5	a+b	a-b	a+d	a-d	b+c	b-c	c+d	c-d
2	6	a+b	a-b	a+d	a-d	b+c	b-c	c+e	c-e
3	7,8,9,10	a+b	a-b	a+d	a	b+c	b-c	c	c-d
4	11,12	a+b	a-b	a+2c	a	b+c	b-c	c	2a+c
5	13,14	a+b	a-b	a+b+c	a+b-c	b+c	b-c	a-b+c	a-b-c
6	15,16,17,18 31,32,33,34	a	3a	5a	7a	9a	11a	13a	21a
7	19,20,21,22	a+b	b	2a+b	2a-b	3a+b	3a-b	4a-b	5a-b
8	23,24,25,26	a+b	a-b	3a+b	3a-b	4a+b	4a-b	5a+b	5a-b
9	27,28,29,30	a	7a	b	2a+b	2a-b	4a+b	4a-b	6a+b

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